# *Mechanics of Materials*

**(http://bernoulli.iam.ntu.edu.tw/)**

## Chapter 7

# Flexural (彎曲) Loading:
# Stresses in Beams

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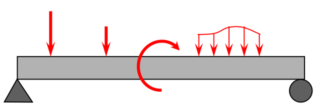
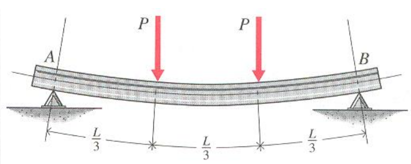
# 7-1 Introduction

- Beam: a long member subjected to transverse loading



flexural member


*Bending a glass plate* ?

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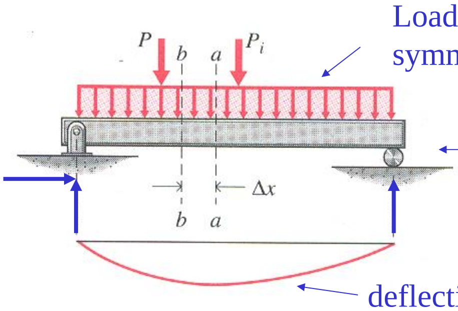
# Classification (I)

- Simple beam (statically determinate)  簡支樑

Simply supported beam

Loads act in the plane of symmetry.




Supported by
pin, roller,
smooth surface

deflection curve

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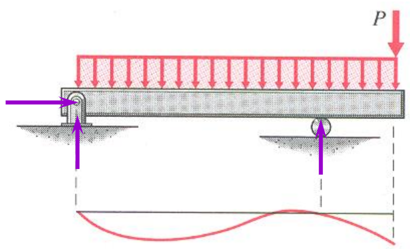
# Classification (II)

- Simple beam with overhang (statically determinate)



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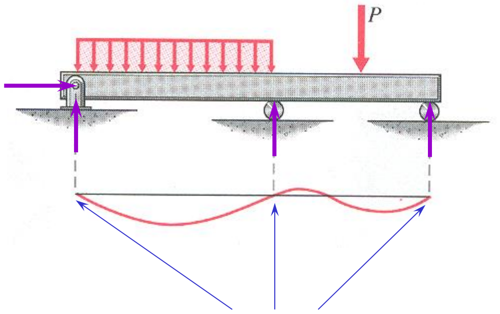
## Classification (III)

- Continuous beam (statically indeterminate)




more than two simple supports,
no movements at the supports

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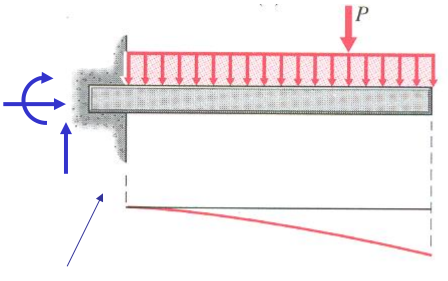
## Classification (IV)

- Cantilever beam (statically determinate) 懸臂梁




built-in end
fixed (no rotation occurs)
or restrained (a limited rotation occurs)

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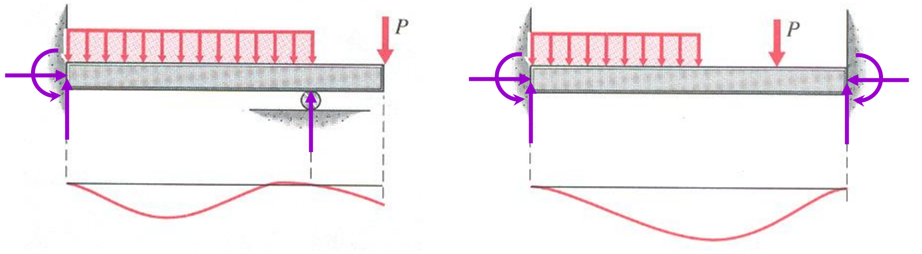
## Classification (V)

- Other types (statically indeterminate)




(Fixed end: no movement, no rotation
Restrained end: no movement, limited rotation)

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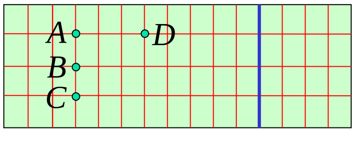
## Experiment and Discussion

- Draw grid lines on the rubber




- Bend the rubber and observe its behavior.
  - deformed shape of cross-section
  - elongation of logitudinal fibers
- Can you predict the strains and stresses at point *A*, *B*, *C*, and *D*?

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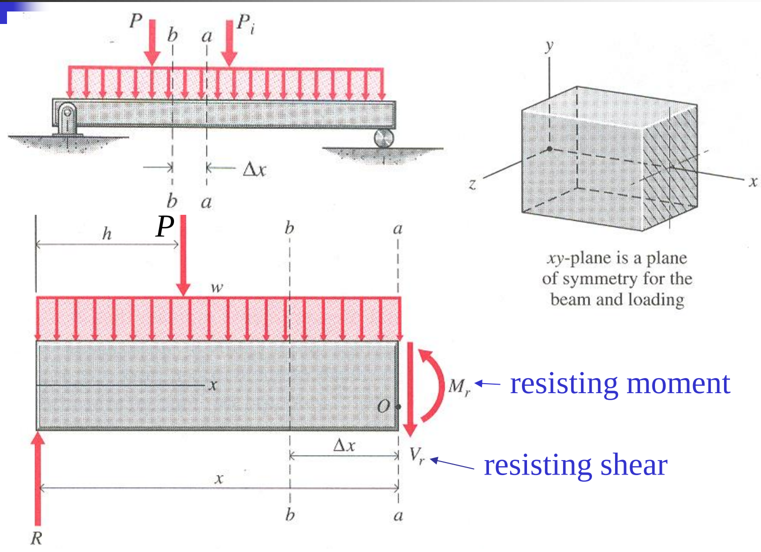
## Resisting Moment and Shear

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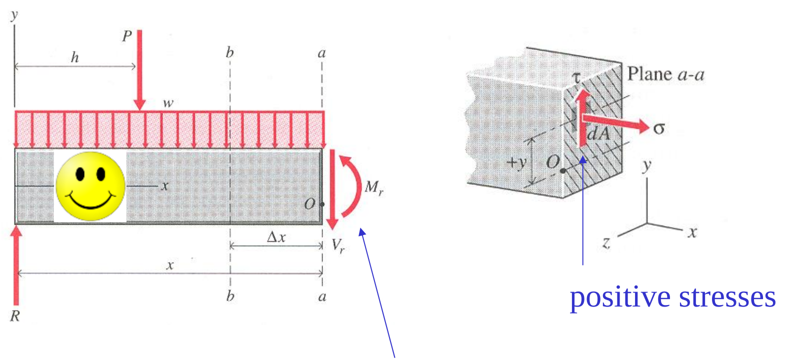
## Resisting Moment and Shear

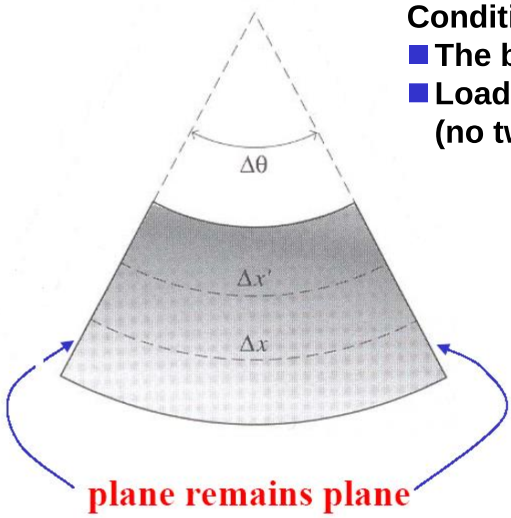



$$V_r = -\int_{\text{area}} \tau \, dA$$

$$M_r = -\int_{\text{area}} y \, \sigma \, dA$$

**positive** resisting moment $M_r$
and **shear force** $V_r$

## 7-2 Flexural Strains (I)




**Conditions:**
- **The beam is bent with couples**
- **Loading in plane of symmetry (no twisting occurs)**

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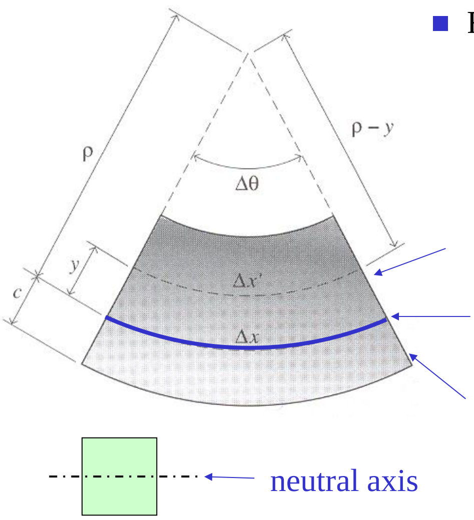
## 7-2 Flexural Strains (II)




- Fiber: a longitudinal element

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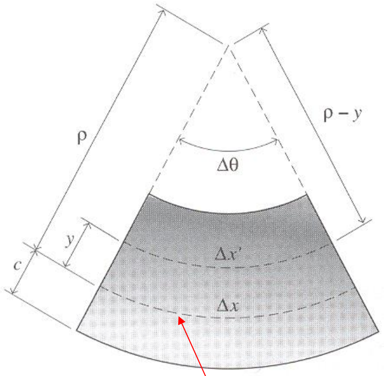
## 7-2 Flexural Strains (III)

- Assume: All fibers have the same initial length.

  ⇨ **prismatic beam**




neutral surface

$$\varepsilon_x = \frac{\delta}{L} = \frac{L_f - L_i}{L_i}$$

$$= \frac{\Delta x' - \Delta x}{\Delta x} = \frac{(\rho - y)(\Delta\theta) - \rho(\Delta\theta)}{\rho(\Delta\theta)}$$

$$= -\frac{1}{\rho} y$$

valid for elastic or inelastic action

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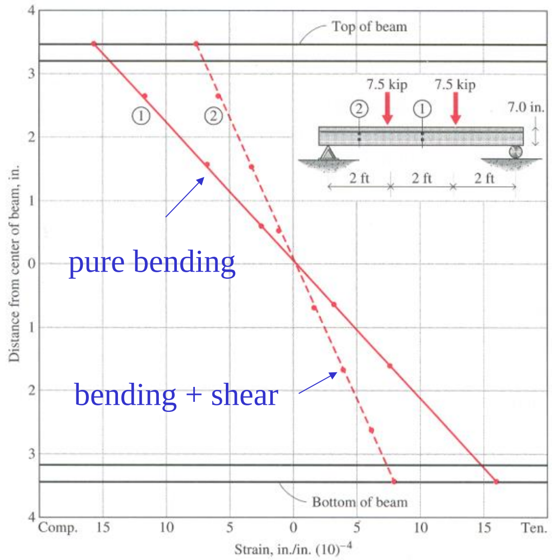
## 7-2 Flexural Strains (IV)




pure bending

bending + shear

$\varepsilon_x \propto y$

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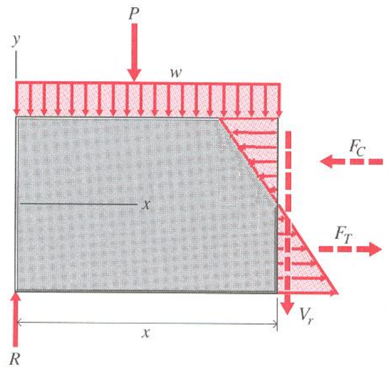
## 7-3 Flexural Stresses (I)

$$\varepsilon_x = -\frac{1}{\rho} y$$

$$\sigma_x = E\varepsilon_x = -\frac{E}{\rho} y$$

uniaxial stress
valid for pure bending

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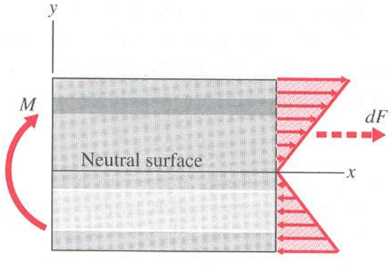
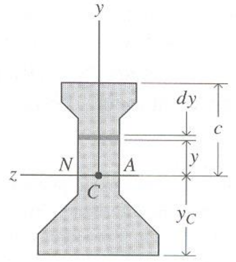
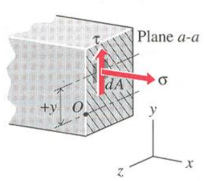
## 7-3 Flexural Stresses (II)

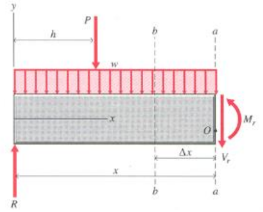



$$M_r = -\int_A y\, dF = -\int_A y\, \sigma_x\, dA$$

Where is the neutral surface?



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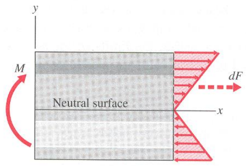
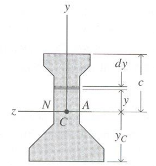
# 7-3 Flexural Stresses (III)

- Location of Neutral Axis (N.A.)



$$\sum F_x = \int_A \sigma_x \, dA = 0$$

$$\int_A \sigma_x \, dA = \int_A \left( E(-\frac{y}{\rho}) \right) dA = -\frac{E}{\rho} \int_A y \, dA = -\frac{E}{\rho} y_c A = 0$$

$y_c$: distance from the reference axis (neutral axis) to the centroidal axis *c*-*c* of the cross section

The N.A. passes through the centroid of the cross section for linearly elastic action

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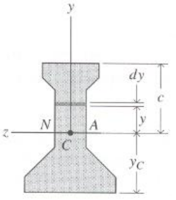
# 7-3 Flexural Stresses (IV)

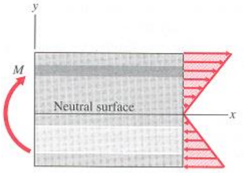


$$\sigma_x = -\frac{E}{\rho} y$$

$$\sigma_{max} = -\frac{E}{\rho} c$$

farthest distance from N.A. to the surface

$$\sigma_x = \frac{y}{c} \sigma_{max} = \frac{y}{c} \sigma_c$$




$$M_r = -\int_A y \, \sigma_x \, dA = -\frac{\sigma_c}{c} \int_A y^2 dA$$

**I**: second moment of area (see Appendix A, A11, Table A-1,)

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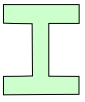
## 7-4 The Elastic Flexural Formula

$$M_r = -\frac{\sigma_c}{c}\int_A y^2 dA = -\frac{\sigma_c I}{c}$$

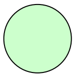
$$\sigma_c = -\frac{M_r c}{I}$$

$$\sigma_x = \frac{y}{c}\sigma_c = -\frac{M_r y}{I} \qquad \boxed{\sigma_x(y) = -\frac{M_r y}{I}}$$

$$\sigma_{\max} = -\frac{M_r c}{I} = -\frac{M_r}{S} \qquad S = \frac{I}{c}$$ **section modulus** of the beam



$S_{\text{I-beam}} > S_{\text{Circle}}$



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## Nonsymmetric Sections

- The flexure formula can be applied to nonsymmetric sections if

$$\int_A z\,\sigma_x dA = \int_A z\frac{\sigma_c}{c}ydA = \frac{\sigma_c}{c}\int_A zy\,dA = \frac{\sigma_c}{c}I_{yz} = 0$$

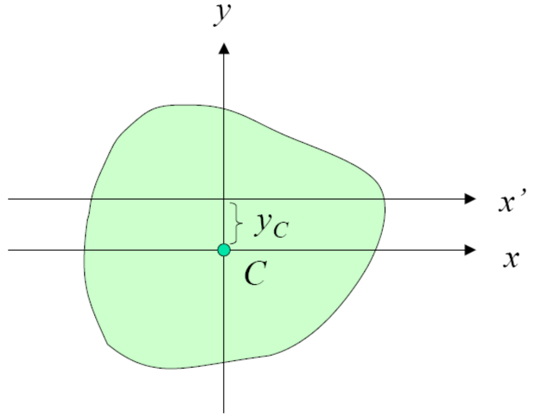
- $I_{yz}$: mixed second moment of the cross sectional area with respect to the centroidal *y* and *z*-axes.
- $I_{yz} = 0$ if
  - either *y*- or *z*-axis is an axis of symmetry (symm. Section)
  - *y*- and *z*-axes are centroidal principal axes (see p. A 17, Appendix A-2-5)

## Parallel Axis Theorem

$$\boxed{I_{x'} = I_{xC} + Ay_C^2}$$

$I_{x'}$ : $I$ w.r.t. to $x'$ axis

$I_{xC}$ : $I$ w.r.t. to the centroid

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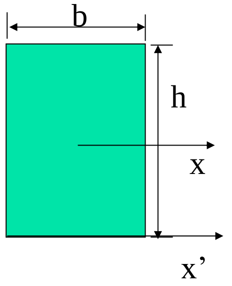
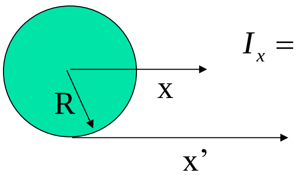
## Second Moments of plane areas




$$I_x = \frac{bh^3}{12}$$

$$I_{x'} = \frac{bh^3}{3}$$




$$I_x = \frac{\pi R^4}{4} \qquad I_{x'} = \frac{5\pi R^4}{4}$$

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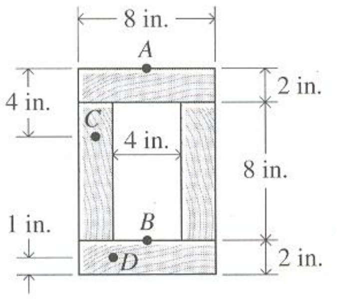
## Example Problem 7-1 (I)




- $M_r = -200(10^3)$ in.·lb

Determine

- $\sigma_A = ?$
- $\sigma_B = ?$
- $\sigma_C = ?$
- $\sigma_D = ?$

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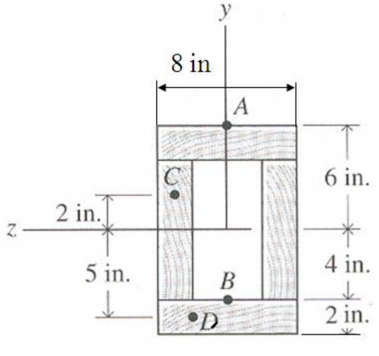
## Example Problem 7-1 (II)

- Symmetric section ⇨ centroid = geometric center




$$y_A = +6\,\text{in.}$$

$$y_B = -4\,\text{in.}$$

$$y_C = +2\,\text{in.}$$

$$y_D = -5\,\text{in.}$$

$$I = \frac{b_s h_s^3}{12} - \frac{b_h h_h^3}{12} = \frac{8(12)^3}{12} - \frac{4(8)^3}{12} = 981.3\,\text{in.}^4$$

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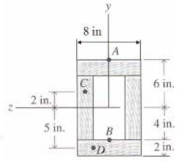


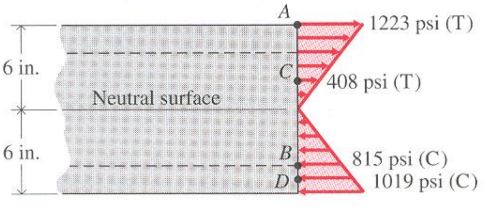
## Example Problem 7-1 (III)

$$\sigma_A = -\frac{M_r y_A}{I} = -\frac{-200(10^3)(+6)}{981.3} = +1222.9\ lb/in.^2 \cong 1223\ psi\,(T)$$

$$\sigma_B = -\frac{M_r y_B}{I} = -\frac{-200(10^3)(-4)}{981.3} = -815.2\ lb/in.^2 \cong 815\ psi\,(C)$$

$$\sigma_C = -\frac{M_r y_C}{I} = -\frac{-200(10^3)(+2)}{981.3} = +407.6\ lb/in.^2 \cong 408\ psi\,(T)$$

$$\sigma_D = -\frac{M_r y_D}{I} = -\frac{-200(10^3)(-5)}{981.3} = -1019.1\ lb/in.^2 \cong 1019\ psi\,(C)$$

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## Example Problem 7-1 (IV)

- Alternatively,

$$\sigma_x = -\frac{M_r y}{I} \Rightarrow \frac{\sigma_x}{y} = -\frac{M_r}{I} = \text{constant}$$

$$\Rightarrow \frac{\sigma_A}{y_A} = \frac{\sigma_B}{y_B} = \frac{\sigma_C}{y_C} = \frac{\sigma_D}{y_D}$$

$$\sigma_B = \frac{y_B}{y_A}\sigma_A = \frac{-4}{+6}(1222.9) = -815.3\ \text{lb/in.}^2 \cong 815\ \text{psi (C)}$$

$$\Rightarrow \sigma_C = \frac{y_C}{y_A}\sigma_A = \frac{+2}{+6}(1222.9) = +407.6\ \text{lb/in.}^2 \cong 408\ \text{psi (T)}$$

$$\sigma_D = \frac{y_C}{y_A}\sigma_A = \frac{-5}{+6}(1222.9) = -1019.1\ \text{lb/in.}^2 \cong 1019\ \text{psi (C)}$$

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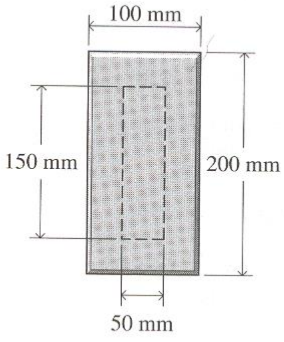
## Example Problem 7-2 (I)




- $\sigma_{max}$ = 15 MPa

*Determine:*

- $M_r$ = ?
- percentage decrease in $M_r$ = ? if the dotted central protion is removed.

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## Example Problem 7-2 (II)

- Symmetric section ⇨ centroid = geometric center

$$\sigma_x = -\frac{M_r y}{I} \qquad M_r = -\frac{\sigma_x I}{y}$$

$$I = \frac{100(200)^3}{12} = 66.67\left(10^6\right)\text{mm}^4 = 66.67\left(10^{-6}\right)\text{m}^4$$

$$\left|M_r\right| = \frac{15\left(10^6\right)(66.67)\left(10^{-6}\right)}{100\left(10^{-3}\right)} = 10.00\left(10^3\right)\text{N}\cdot\text{m} = 10.00\ \text{kN}\cdot\text{m}$$

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## Example Problem 7-2 (III)

- Hollow section

$$I = \frac{100(200)^3}{12} - \frac{50(150)^3}{12} = 52.60\left(10^6\right)\text{mm}^4 = 52.60\left(10^{-6}\right)\text{m}^4$$

$$|M_r| = \frac{\sigma_x I}{c} = \frac{15\left(10^6\right)(52.60)\left(10^{-6}\right)}{100\left(10^{-3}\right)} = 7.89\left(10^3\right)\text{N}\cdot\text{m} = 7.89\ \text{kN}\cdot\text{m}$$

$$\text{Percent decrease} = D = \frac{10.00 - 7.89}{10.00}100 = 21.1\%$$

Note: $A$ –38% near N.A. $M_r$ – 21%
$A$ –38% from top and bottom $M_r$ – 61%

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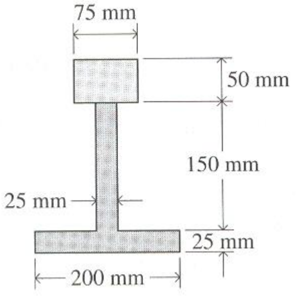
## Example Problem 7-3 (I)




- $M_r$ = –75 kN·m

Determine:

- $\sigma_{max}$ (Tensile) = ?
- $\sigma_{max}$ (compressive) = ?

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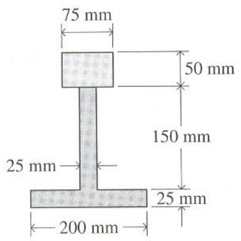
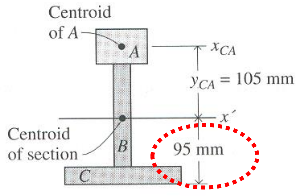
## Example Problem 7-3 (II)

- Locate centroid




$$A = 200(25) + 150(25) + 50(75) = 12{,}500\ \text{mm}^2$$

$$M_A(\text{w.r.t. bottom}) = 200(25)(12.5) + 25(150)(100) + 75(50)(200)$$
$$= 1{,}187{,}500\ \text{mm}^3$$

$$y_c = \frac{M_A}{A} = \frac{1{,}187{,}500}{12{,}500} = 95\ \text{mm}$$

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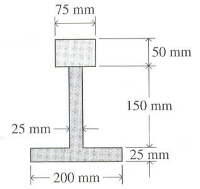
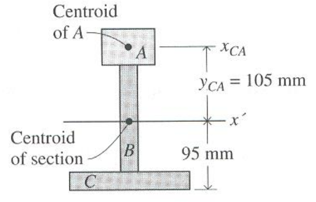
## Example Problem 7-3 (III)




- Compute $I_{x'}$ (parallel axis theorem)




$I_{x'} = I_{xC} + Ay_C^2$ $\quad$ $I_{x'}$ : $I$ w.r.t. to $x'$ axis

$I_{xC}$ : $I$ w.r.t. to the centroid

$$\boldsymbol{I_{x'A} = I_{xCA} + A_A y_{CA}^2} = \frac{75(50)^3}{12} + 75(50)(105)^2 = 42.13(10^6)\ \text{mm}^4$$

$$\boldsymbol{I_{x'B} = I_{xCB} + A_B y_{CB}^2} = \frac{25(150)^3}{12} + 25(150)(5)^2 = 7.13(10^6)\ \text{mm}^4$$

$$\boldsymbol{I_{x'C} = I_{xCC} + A_C y_{CC}^2} = \frac{200(25)^3}{12} + 200(25)(-82.5)^2 = 34.29(10^6)\ \text{mm}^4$$

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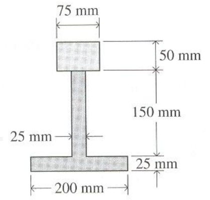
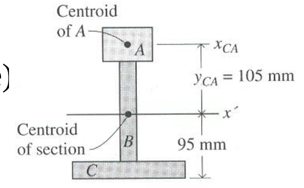
## Example Problem 7-3 (IV)




- Compute $I_{x'}$

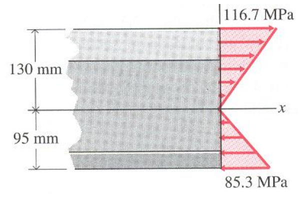
$$I_{x'} = I_{x'A} + I_{x'B} + I_{x'C} = 42.13\left(10^6\right) + 7.13\left(10^6\right) + 34.29\left(10^6\right)$$
$$= 83.55\left(10^6\right)\text{mm}^4 = 83.55\left(10^{-6}\right)\text{m}^4$$




- Compute $\sigma_{max}$ (Tensile) and $\sigma_{max}$ (Compressive)

$$\sigma_{max}(\text{T}) = -\frac{M_r y_t}{I} = -\frac{-75\left(10^3\right)(130)\left(10^{-3}\right)}{83.55\left(10^{-6}\right)}$$

$\cong 116.7$ MPa (T)




$$\sigma_{max}(\text{C}) = -\frac{M_r y_b}{I} = -\frac{-75\left(10^3\right)(-95)\left(10^{-3}\right)}{83.55\left(10^{-6}\right)}$$

$\cong 85.3$ MPa (C)

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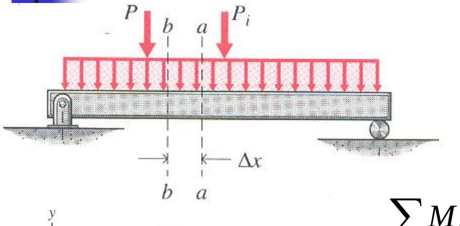
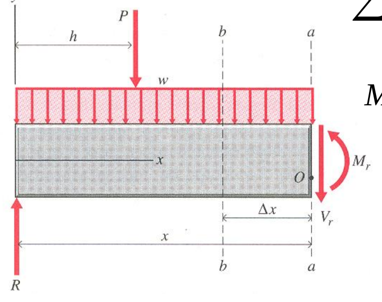
## 7-5 Shear Forces and Bending Moments in Beams




$$\sum F_y = R - wx - P - V_r = 0$$

$$V_r = R - wx - P = V$$



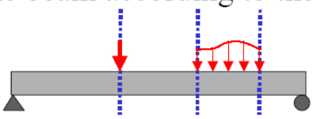

$$\sum M_O = -Rx + \frac{wx^2}{2} + P(x-h) + M_r = 0$$

$$M_r = Rx - \frac{wx^2}{2} - P(x-h) = M$$

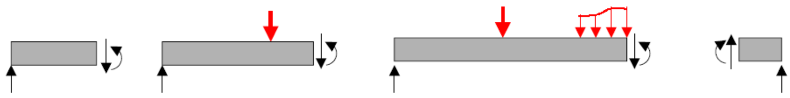
$V_r$, $M_r$: resisting force and moment
$V$, $M$: resultant force and moment at the given section

## Procedure to determine M(x) and V(x)

- Divide the beam according to the loads.



- For each section, draw a FBD.



- Apply equilibrium equations to each section.

$$\sum F_y = 0 \qquad \Rightarrow \quad V(x)$$

$$\sum M_{\text{cut}} = 0 \qquad \Rightarrow \quad M(x)$$

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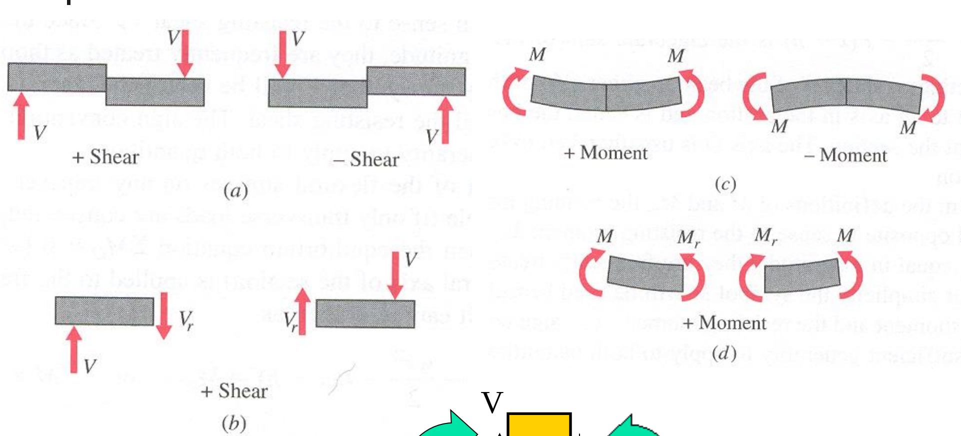
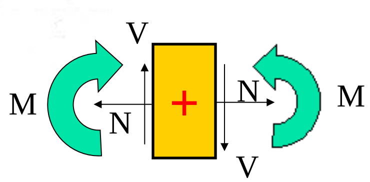
## Sign Convention



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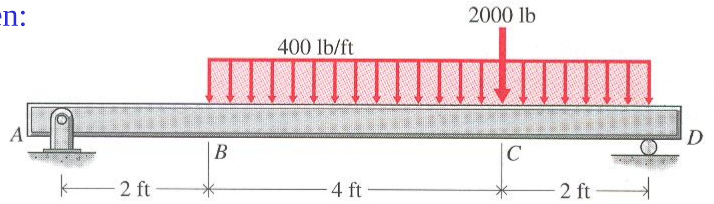
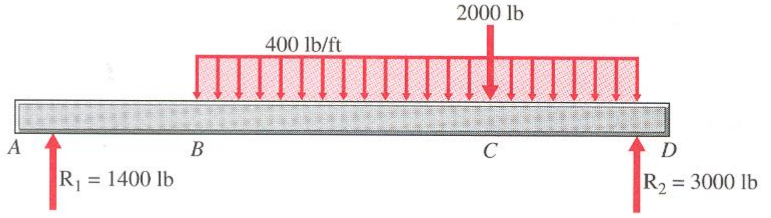
## Example Problem 7-5 (I)

Given:




- $\Sigma M_D = 0$, $\Sigma M_A = 0$ ⇨ $R_1 = 1400$ lb and $R_2 = 3000$ lb

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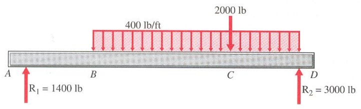
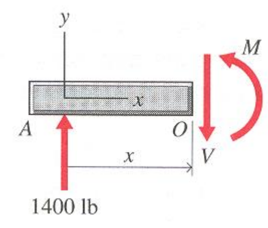
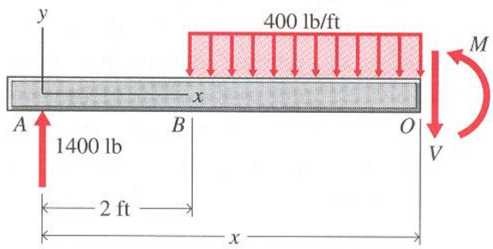
## Example Problem 7-5 (II)






*AB*: $0 < x < 2$

$$\sum F_y = 1400 - V = 0$$

$$\boxed{V = 1400 \text{ lb}}$$

$$\sum M_O = -1400x + M = 0$$

$$\boxed{M = 1400x \text{ ft} \cdot \text{lb}}$$

*BC*: $2 < x < 6$

$$\sum F_y = 1400 - 400(x-2) - V = 0$$

$$\boxed{V = -400x + 2200 \text{ lb}}$$

$$\sum M_O = -1400x + 400(x-2)\left(\frac{x-2}{2}\right) + M = 0$$

$$\boxed{M = -200x^2 + 2200x - 800 \text{ ft} \cdot \text{lb}}$$

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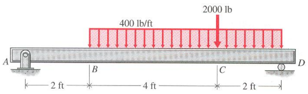
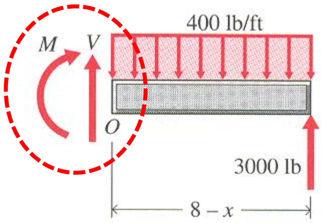
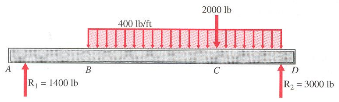



## Example Problem 7-5 (III)

*CD*: $6 < x < 8$




$$\sum F_y = V - 400(8-x) + 3000 = 0 \qquad \boxed{V = -400x + 200 \text{ lb}}$$

$$\sum M_O = -M - 400(8-x)\left(\frac{8-x}{2}\right) + 3000(8-x) = 0$$

$$\boxed{M = -200x^2 + 200x + 11{,}200 \text{ ft}\cdot\text{lb}}$$

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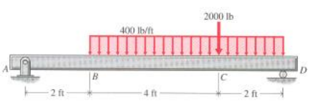
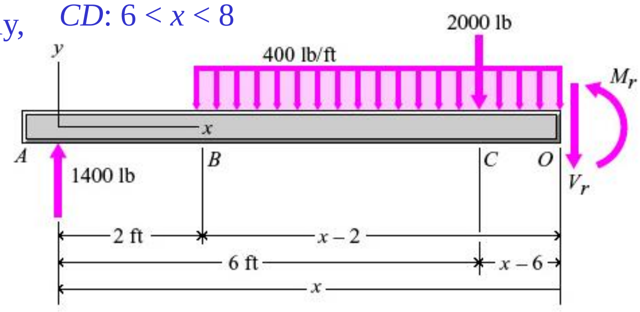



## Example Problem 7-5 (IV)

Alternatively, *CD*: $6 < x < 8$




$$\sum F_y = 1400 - 400(x-2) - 2000 - V_r = 0 \qquad \boxed{V_r = -400x + 200 \text{ lb}}$$

$$\sum M_O = -1400x + 400(x-2)\left(\frac{x-2}{2}\right) + 2000(x-6) + M_r = 0$$

$$\boxed{M_r = -200x^2 + 200x + 11{,}200 \text{ ft}\cdot\text{lb}}$$

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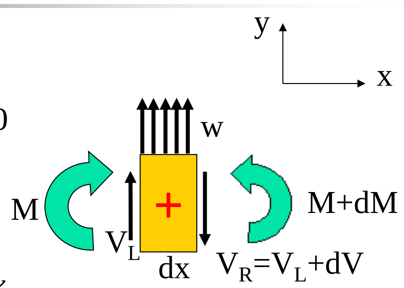
# 7-6 Shear Forces and Bending Moments in Beams

- If distributed load exists




$$\sum F_y = \cancel{V_L} + w dx - (\cancel{V_L} + dV) = 0$$

$$\boxed{\frac{dV}{dx} = w}$$

$$V_2 - V_1 = \int_{V_1}^{V_2} dV = \int_{x_1}^{x_2} w\,dx$$

$$\sum M_L = 0$$

$$-\cancel{M} + \cancel{wdx\left(\frac{dx}{2}\right)} - (V_L + \cancel{dV})dx + (\cancel{M} + dM) = 0$$

$$\Rightarrow \boxed{\frac{dM}{dx}\Big|_L = V_L}$$

How about using $\sum M_R = 0$ ?
Please do by yourself.

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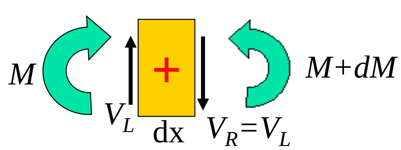
# 7-6 Shear Forces and Bending Moments in Beams

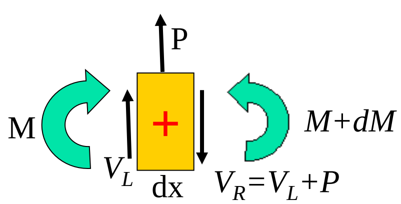
- No concentrated load or distributed load




$$\frac{dV}{dx} = 0 \qquad \Delta V = V_R - V_L = 0$$

$$\frac{dM}{dx} = V_L = V_R \quad \text{or} \quad \boxed{\frac{dM}{dx} = V}$$

- If concentrated load exists




$$V_R = V_L + P$$

$$\Delta V = V_R - V_L = P$$

$$\left.\frac{dM}{dx}\right)_L = V_L \qquad \left.\frac{dM}{dx}\right)_R = V_R$$

$$M_2 - M_1 = \int_{M_1}^{M_2} dM = \int_{x_1}^{x_2} V\,dx$$

***Change of Moment = area under V-diagram***

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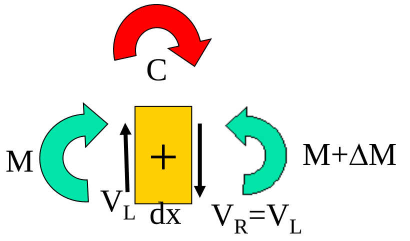
# 7-6 Shear Forces and Bending Moments in Beams

- If concentrated couple exists, $C \neq 0$

$$\Delta V = V_R - V_L = 0$$

$$\Delta M = C$$

$$M_R = M_L + C$$

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# Shear and Bending Moment Diagrams

Dictate variations of shear and bending moment.

- Method 1 (代數法):
  - Establish algebraic equations for shear forces and bending moments.
  - Construct curves from the equations.
- Method 2 **(圖解法，Convenient)** :
  - Draw the free body diagram for the entire beam with applied loads and reactions on proper coordinates.
  - Construct the shear diagram directly below the FBD.
  - Plot the bending moment diagram further below.

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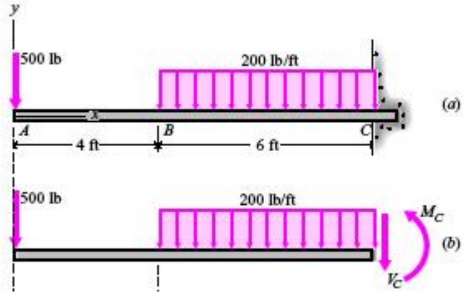
## Example Problem 7-7

Given:




Find:

- Write $V(x)$ and $M(x)$ in the interval $AB$
- Write $V(x)$ and $M(x)$ in the interval $BC$
- Draw $V$ and $M$ diagrams

Sol: $\sum F_y = 0: \quad -500 - 200 \cdot 6 - V_C = 0$

$$V_C = -1700 \text{ lb}$$

$\sum M_C = 0: \quad 500 \cdot 10 + 200 \cdot 6 \cdot 3 + M_C = 0$

$$M_C = -8600 \text{ lb} \cdot \text{ft}$$

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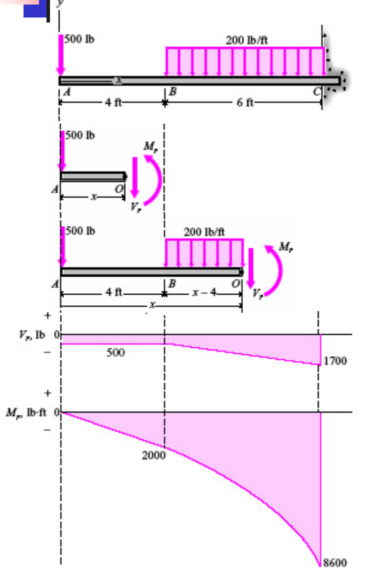
## Example Problem 7-7

$$V_C = -1700 \text{ lb}$$
$$M_C = -8600 \text{ lb} \cdot \text{ft}$$




*AB*: $0 < x < 4$ ft

$\sum F_y = 0: \quad -500 - V_r = 0$

$$V_r = -500 \text{ lb}$$

$\sum M_O = 0: \quad 500 \cdot x + M_r = 0$

$$M_r = -500 \cdot x \text{ lb} \cdot \text{ft}$$

*BC*: $4 < x < 10$ ft

$\sum F_y = 0: \quad -500 - 200(x-4) - V_r = 0$

$$V_r = 300 - 200x \text{ lb}$$

$\sum M_O = 0: \quad 500 \cdot x + 200(x-4) \cdot (x-4)/2 + M_r = 0$

$$M_r = -100x^2 + 300 \cdot x - 1600 \text{ lb} \cdot \text{ft}$$

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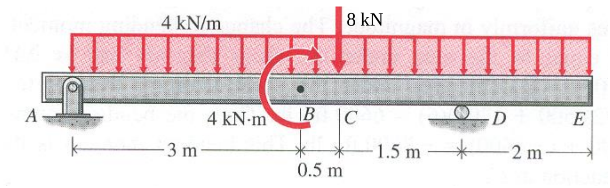
## Example Problem 7-8 (I)

Given:




Find:
- Write equations of $V(x)$ and $M(x)$ in the interval $CD$
- Draw $V$ and $M$ diagrams

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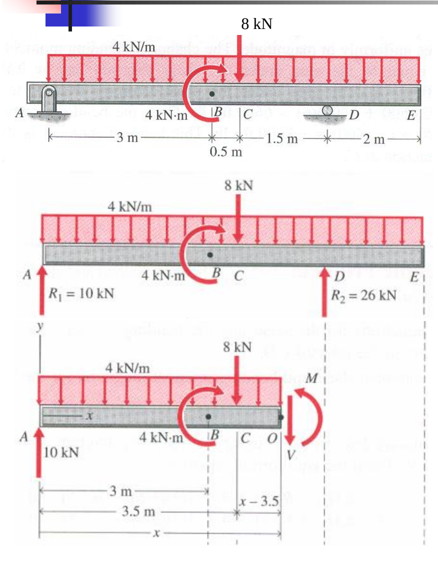
## Example Problem 7-8 (II)




$$\sum M_D = -5R_1 - 4 + 4(7)(1.5) + 8(1.5) = 0$$

$$\sum M_A = 5R_2 - 4 - 4(7)(3.5) - 8(3.5) = 0$$

$$R_1 = 10\text{ kN} \qquad R_2 = 26\text{ kN}$$

$CD$: $3.5 < x < 5$ m

$$\sum F_y = 10 - 4x - 8 - V = 0$$

$$V = -4x + 2\text{ kN}$$

$$\sum M_O = -10(x) + 4x(x/2) - 4 + 8(x - 3.5) + M = 0$$

$$M = -2x^2 + 2x + 32\text{ kN}\cdot\text{m}$$

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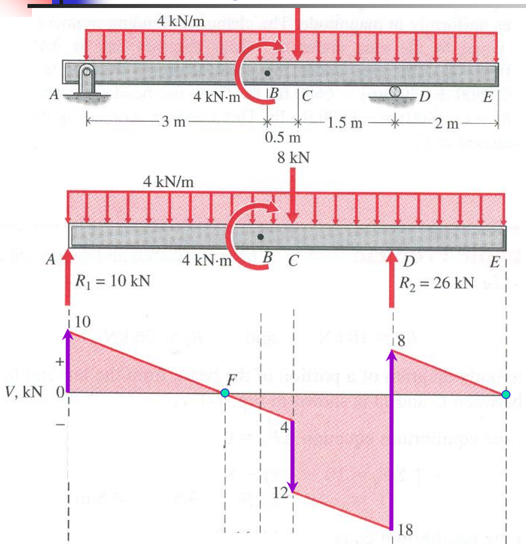
## Example Problem 7-8 (III)




**Draw *V* diagram:**

$$V_C^- = 10 - 4(3.5) = -4\,\text{kN}$$

$$V_C^+ = -4 - 8 = -12\,\text{kN}$$

$$V_D^- = -12 - 4(1.5) = -18\,\text{kN}$$

$$V_D^+ = -18 + 26 = 8\,\text{kN}$$

$$V_E = +8 - 4(2) = 0\,\text{kN}$$

$$x_1 = 10/4 = 2.5\,\text{m}$$

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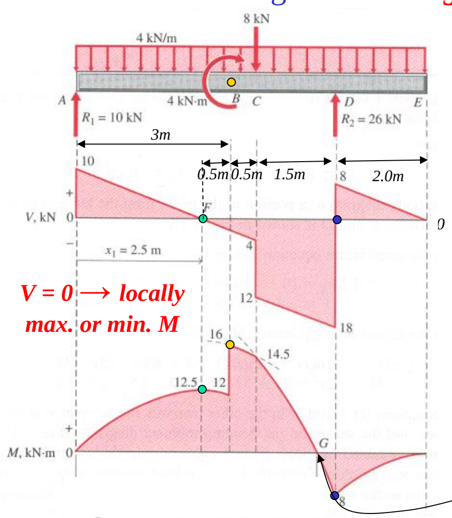
## Example Problem 7-8 (IV)

Draw *M* diagrams ***Change of Moment = area under V-diagram***




***V* = 0 ⟶ locally max. or min. *M***

$$M_F = 0 + 10(2.5)/2 = 12.5\,\text{kN}\cdot\text{m}$$

$$V_B = 10 - 4(3) = -2\,\text{kN}$$

$$M_{B\_} = 12.5 - 2(0.5)/2 = 12.0\ \text{kN}\cdot\text{m}$$

$$M_{B+} = 12.0 + 4 = 16.0\ \text{kN}\cdot\text{m}$$

$$M_C = 16.0 - (4+2)(0.5)/2 = 14.5\ \text{kN}\cdot\text{m}$$

$$M_D = 14.5 - (12+18)(1.5)/2 = -8.0\ \text{kN}\cdot\text{m}$$

$$M_E = -8.0 + 8(2)/2 = 0\ \text{kN}\cdot\text{m}$$

*Solving* $M = -2x^2 + 2x + 32\ kN\cdot m = 0$

$$x_2 = 0.5 \pm \sqrt{16.25} \cong 4.531\ \text{m}$$

*M = 0, for sections to be spliced*

*The **max M** occurs at position B where **V ≠0** in this problem!*

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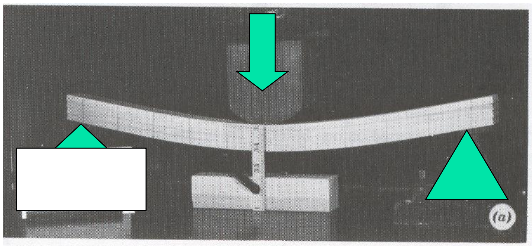
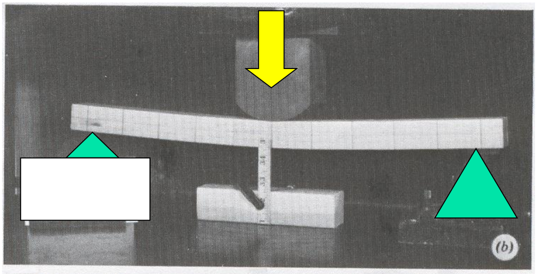
Flexural Stresses: $\sigma_x$

# 7-7 Shearing Stresses in Beams



- Stacking flat slabs (平板) **without gluing**.
- Load: 200 lb
- Deflection: 1.18"
- Note rugged ends

laminated beam



- Stacking flat slabs **with gluing**.
- Load: 800 lb
- Deflection: 0.375"

Why? Glue-laminated beam

shearing between slabs

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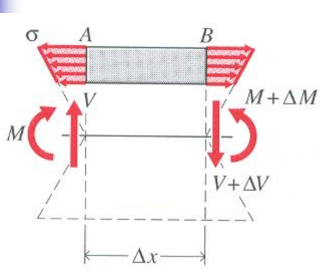
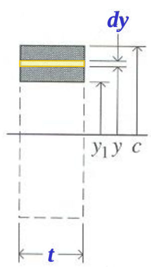
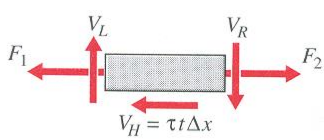
# 7-7 Shearing Stresses in Beams (I)




$$F_1 = -\frac{M}{I}\int_A y\,dA = -\frac{M}{I}\int_{y_1}^{c} y\,(t\,dy)$$

$$F_x = \int_A \sigma_x dA$$

$$\sigma_x = -\frac{My}{I}$$

$$F_2 = -\frac{M+\Delta M}{I}\int_{y_1}^{c} y\,(t\,dy)$$

Force balance: $$V_H = F_2 - F_1 = -\frac{\Delta M}{I}\int_{y_1}^{c} y\,(t\,dy)$$

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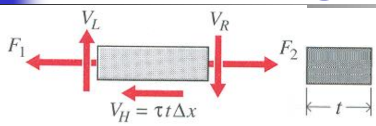
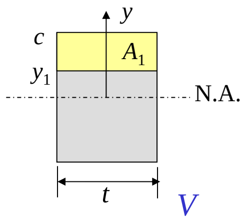
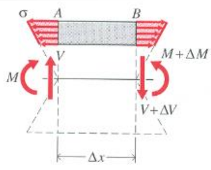
## 7-7 Shearing Stresses in Beams (II)




$$\boldsymbol{V_H = -\frac{\Delta M}{I}\int_{y_1}^{c} y\,(t\,dy)} \qquad \boldsymbol{A_s = t\Delta x}$$

$$\tau_{avg} = \frac{V_H}{A_s} = -\frac{\Delta M}{It\Delta x}\int_{y_1}^{c} y\,(t\,dy)$$

$V$

$$\tau_{(H)} = \lim_{\Delta x \to 0}\frac{V_H}{A_s} = \lim_{\Delta x \to 0}\frac{\Delta M}{\Delta x}\left(-\frac{1}{It}\right)\int_{y_1}^{c} ty\,dy = \frac{dM}{dx}\left(-\frac{1}{It}\right)\int_{y_1}^{c} ty\,dy$$

$Q$ : 1st moment of $A_1$

$$\tau_{(H)} = -\frac{VQ}{It}$$

$$\boxed{\tau = \frac{VQ}{It}}$$

Let sense of τ = sense of $V$

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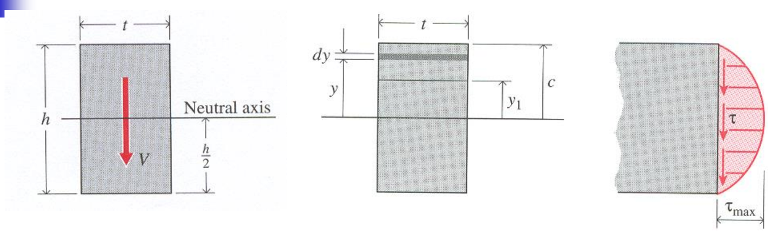
## Shearing Stress in Rectangular Beam




$$\tau = \frac{VQ}{It} = \frac{V}{It}\int_{y_1}^{c} ty\,dy = \frac{V}{I}\int_{y_1}^{h/2} y\,dy = \frac{V}{2I}\left[\left(\frac{h}{2}\right)^2 - y_1^2\right]$$

at $y_1 = 0$ $\qquad \tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8(th^3/12)} = \frac{3}{2}\frac{V}{th} = \frac{3}{2}\frac{V}{A}$

$h = 2t$, error 3%, $h = t$, error 12%, **$h = t/4$, error 100%**

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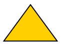
## Remarks

- $\tau = \dfrac{V_r Q}{It}$ is only valid for elastic action.
- $\boldsymbol{\tau}$ is the **average** shearing stress across the thickness *t* and is ***only accurate if t is not too big (wide)***.
- $\tau = \dfrac{V_r Q}{It}$ should ***not*** be applied to a **wide** cross-section, flange of I or T beam, or section where the sides are

  not parallel, e.g., 
- Although the shear stress equation is derived for the horizontal shear area, it is also valid for the vertical transverse section of a beam. $\because \tau_{xy} = \tau_{yx}$

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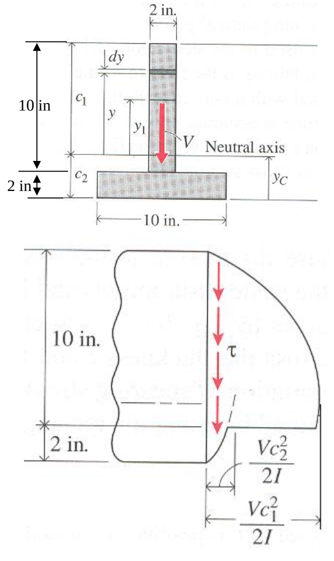
## Shearing Stress in T-shaped Beam




$$y_c = \frac{2(10)(7) + 10(2)(1)}{2(10) + 10(2)} = 4\text{ in.}$$

$$c_1 = 8\text{ in.} \qquad c_2 = 4\text{ in.}$$

$$\tau = \frac{VQ}{It} = \frac{V}{It}\int_{y_1}^{c} ty\,dy$$

$$= \frac{V}{2I}\left(c_1^2 - y_1^2\right) = \frac{V}{2I}\left(8^2 - y_1^2\right) \qquad \text{stem}$$

$$\tau = \frac{V}{2I}\left(c_2^2 - y_1^2\right) = \frac{V}{2I}\left(4^2 - y_1^2\right) \qquad \text{flange}$$

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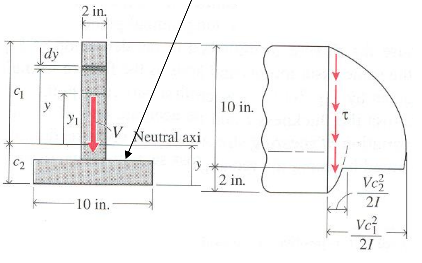
## Shearing Stress in T-shaped Beam

- Stress distribution: parabolic
- Stress discontinuity at the junction of flange and stem.
- Distribution in the flange is fictitious. Flange top should be stress free.
- In general, $\tau_{max}$ occurs at the neutral surface.

  exception: X-shaped beam.

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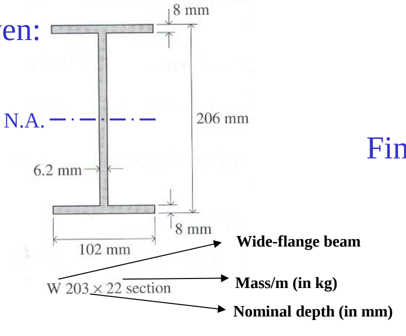
## Shearing Stress in I-Beam

Given:




$V = 37.5$ kN

$I = 20.0(10^6)$ mm$^4$

Find:

- $\tau$ at various points

***Please do it by yourself***

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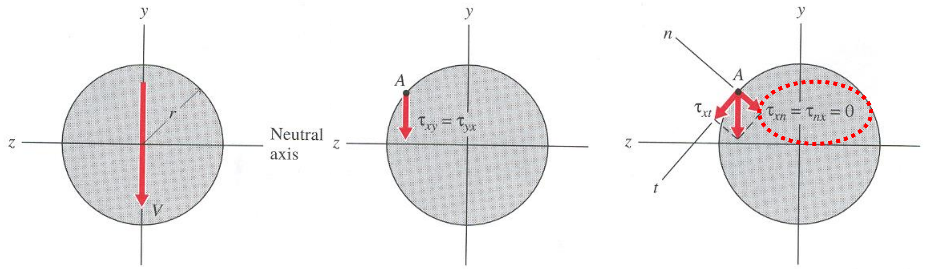
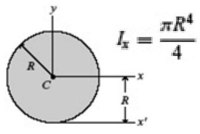
## Shearing Stress in Circular Beam

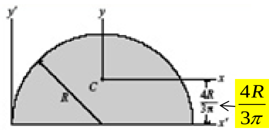



$$\boxed{\tau_{\max}} = \tau_{NA} = \frac{VQ_{NA}}{It_w} = \frac{V\left(\pi r^2/2\right)\left(4r/3\pi\right)}{\left(\pi r^4/4\right)\left(2r\right)} = \frac{4V}{3\pi r^2} = \boxed{\frac{4}{3}\frac{V}{A}}$$

$$\tau_{\max}\left(\text{exact}\right) = 1.38\frac{V}{A}$$

(Table A-1, p. A12; Table A-2, p. A19)

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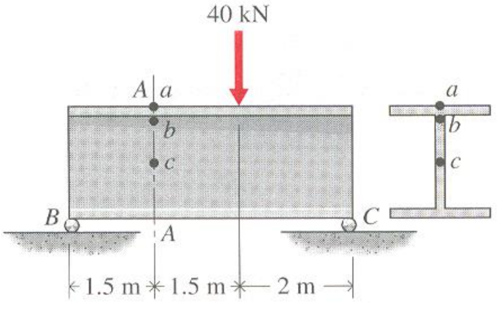
## Example Problem 7-9 (I)




- W254×33 beam

At section *A–A*,

Determine $\sigma$ and $\tau$ at

- Point *a* on the top of the flange
- Point *b* in the web at the junction
- Point *c* on the N.A.

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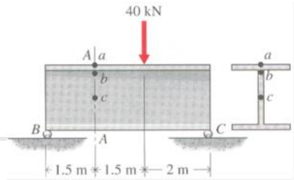
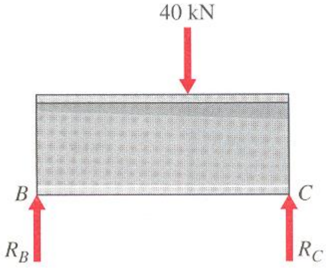
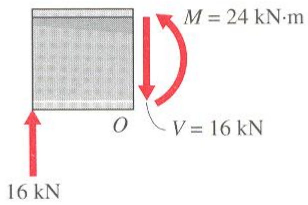
## Example Problem 7-9 (II)




$$\sum M_C = 40(2) - R_B(5) = 0$$
$$\sum M_B = R_C(5) - 40(3) = 0$$
$$R_B = 16\,\text{kN} \qquad R_C = 24\,\text{kN}$$




$$\sum F_y = 16 - V = 0$$
$$\sum M_O = M - 16(1.5) = 0$$
$$V = 16\,\text{kN} \qquad M = 24\,\text{kN}\cdot\text{m}$$

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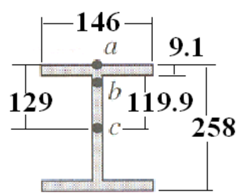
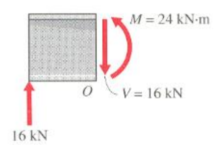
## Example Problem 7-9 (III)

W254×33 section: (p. A27):

$I$ = 49.1($10^6$) $\text{mm}^4$, $d$ (depth) = 258 mm,
$w_f$ = 146mm, $t_f$ = 9.1 mm, $t_w$ = 6.1 mm



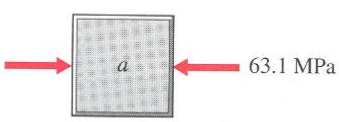

- **Point *a*** (On the top surface)




$$y = d/2 = 129\,\text{mm}$$

$$\sigma_a = -\frac{My}{I} = -\frac{24(10^3)(129)(10^{-3})}{49.1(10^{-6})}\,\text{N/m}^2 \cong 63.1\,\text{MPa}\,(\text{C})$$

$$Q_a = 0$$
$$\tau = 0$$



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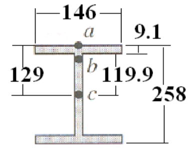
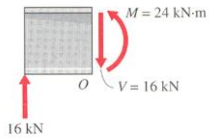
# Example Problem 7-9 (IV)

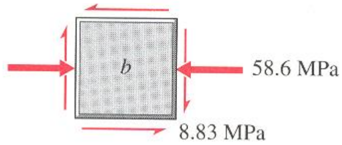



- **Point *b*** (at junction, in the web)

  $I = 49.1(10^6)$ mm$^4$, $d = 258$ mm

  $w_f = 146$mm, $t_f = 9.1$ mm, $t_w = 6.1$ mm




$$y = 129 - 9.1 = 119.9\,\text{mm}$$

$$\sigma_b = -\frac{My}{I} = -\frac{24(10^3)(119.9)(10^{-3})}{49.1(10^{-6})}\,\text{N/m}^2 \cong 58.6\,\text{MPa(C)}$$

$$Q_b = y_C A = (119.9 + 9.1/2)(146)(9.1)\ \text{mm}^3 = 165.34(10^{-6})\ \text{m}^3$$

$$\tau_b = \frac{VQ_b}{I_{NA}t_w} = \frac{16(10^3)(165.34)(10^{-6})}{49.1(10^{-6})(6.1)(10^{-3})}\,\text{N/m}^2 \cong 8.83\,\text{MPa}$$

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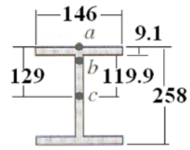
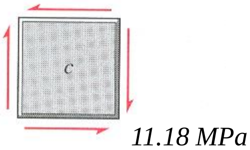
# Example Problem 7-9 (V)




- Point *c* (on N.A.)

  $I = 49.1(10^6)$ mm$^4$, $d = 258$ mm

  $w_f = 146$mm, $t_f = 9.1$ mm, $t_w = 6.1$ mm




$$y = 0 \qquad \sigma_c = 0$$

$$Q_c = y_c A = 124.45(146)(9.1) + 59.95(119.9)(6.1)\ \text{mm}^3 = 209.2(10^{-6})\ \text{m}^3$$

$$\tau_c = \frac{VQ_c}{I_{NA}t_w} = \frac{16(10^3)(209.2)(10^{-6})}{49.1(10^{-6})(6.1)(10^{-3})}\,\text{N/m}^2 \cong 11.18\,\text{MPa}$$

$$\tau_{ave} \cong \frac{V}{A_{web}} = \frac{16(10^3)}{6.1(10^{-3})(239.8)(10^{-3})}\,N/m^2 \cong 10.94\,MPa$$ (approx.)

*(neglect flange area)*

$$D = (11.176 - 10.938)/11.176(100) = 2.13\%$$

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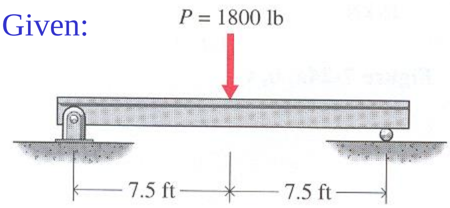
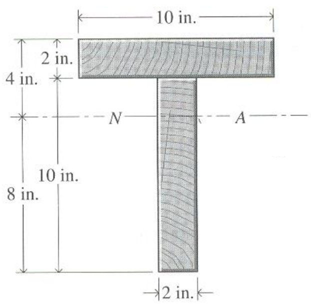
# Example Problem 7-10 (I)

Given:




Determine

- $\tau_{avg}$ on a plane 4 in. above bottom and at $x$ = 6 ft
- $\tau_{max}$ in the beam
- $\tau_{avg}$ in the joint between flange and stem and at $x$ = 6 ft
- Force between flange and stem by the glue in a 12-in. length centered at $x$ = 6 ft
- $\sigma_{max}$ in the beam

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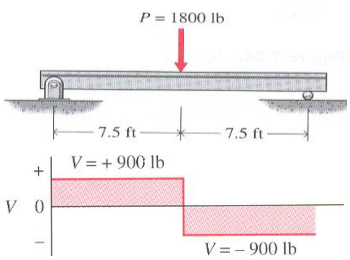
# Example Problem 7-10 (II)




$$V(x = 6\,\text{ft}) = 900\,\text{lb}$$

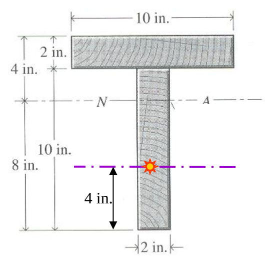
$$I_{NA} = \frac{1}{12}(2)(10)^3 + 2(10)(3)^2 + \frac{1}{12}(10)(2)^3 + 10(2)(3)^2 = 533.3\ \text{in.}^4$$

- $\tau_{avg}$ on a plane 4 in. above bottom and at $x$ = 6 ft

$$Q_4 = y_{C4}A_4 = 6(2)(4) = 48\,\text{in.}^3$$

$$\tau_4 = \frac{VQ_4}{I_{NA}t_4} = \frac{900(48)}{533.3(2)} \cong 40.5\,\text{psi}$$

(sense of $\tau$ = sense of $V$)



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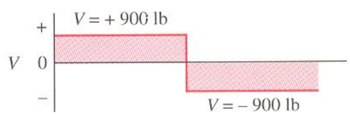
## Example Problem 7-10 (III)

- $\tau_{max}$ in the beam (occurs at N.A.)

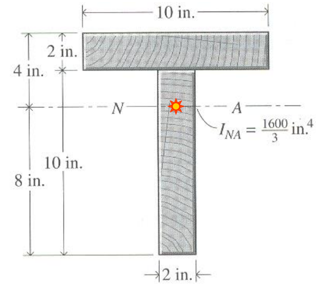
$$V_{max} = 900\ \text{lb}$$




$$Q_{NA} = 4(2)(8) = 64\ \text{in.}^3 \quad \text{(lower)}$$

or $$Q_{NA} = 3(10)(2) + 1(2)(2) = 64\ \text{in.}^3 \quad \text{(upper)}$$

$$\tau_{max} = \frac{VQ_{NA}}{I_{NA}t_s} = \frac{900(64)}{533.3(2)} \cong 54.0\ \text{psi}$$

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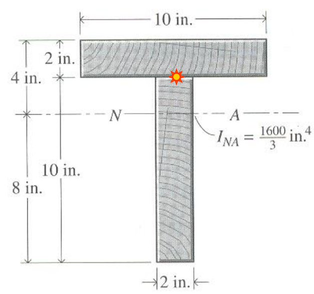
## Example Problem 7-10 (IV)

- $\tau_{avg}$ in the joint between flange and stem and at $x$ = 6 ft

$$Q_F = y_{CF}A_F = 3(10)(2) = 60\ \text{in.}^3$$

$$\tau_J = \frac{VQ_F}{I_{NA}t_s} = \frac{900(60)}{533.3(2)} \cong 50.6\ \text{psi}$$




- Force between flange and stem by the glue (12 in-length)

$$V_g = \tau_J A_J = 50.63(12)(2) \cong 1215\ \text{lb}$$

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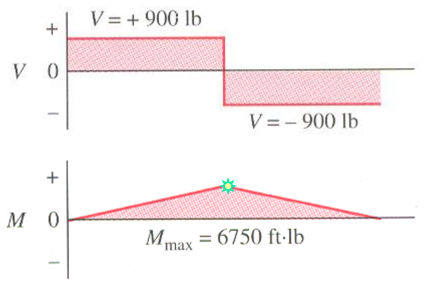
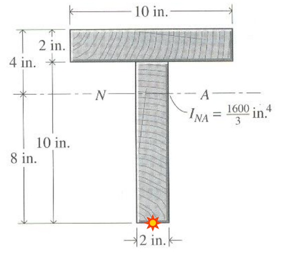
## Example Problem 7-10 (V)

- $\sigma_{max}$ in the beam ($M_{max}$, $c_{max}$)




$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{6750(12)(8)}{533.3} = 1215.1\,\text{psi} \cong 1215\,\text{psi}\,(\text{T})$$

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## 7-8 Principal Stresses in Flexural Members (I)

- On the top or bottom edge of a section

  $\rightarrow \sigma$ is maximum, $\tau = 0$

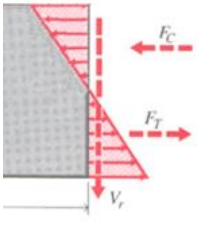
  $\rightarrow \sigma_p = \sigma$, $\tau_p = (\sigma_p - 0)/2$

- At the neutral axis

  $\rightarrow \sigma = 0$, $\tau$ is maximum

  $\rightarrow \sigma_p = \tau_p = \tau$

- What about other points?

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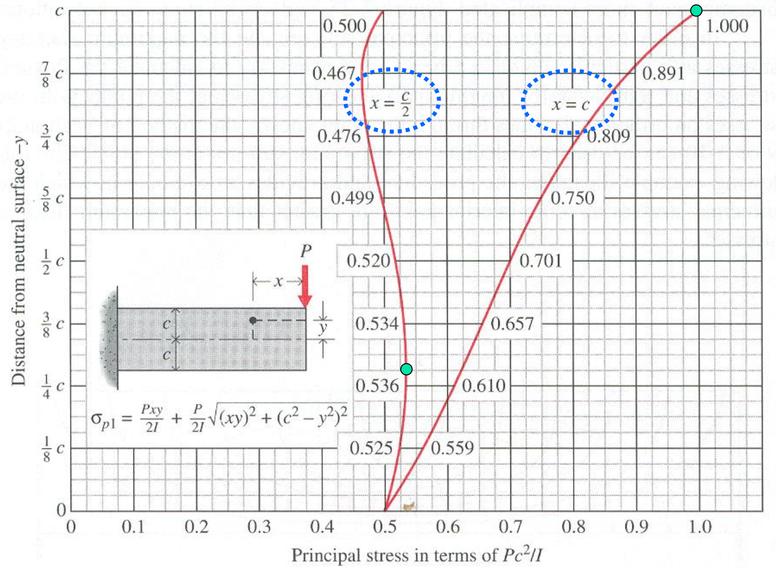
# 7-8 Principal Stresses in Flexural Members (II)

Departure from the flexural formula due to **Saint-Venant's principle**

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# 7-8 Principal Stresses in Flexural Members (III)

- For a section too close to $P$, the flexure formula doesn't apply. (Saint Venant's principle) $\boxed{\sigma_x(y) = -\frac{M_r y}{I}}$
- For a rectangular cross section, in regions where the flexure formula applies,

  max. normal stress = max. flexural stress

  $\tau_{max}$ = ½ (max. normal stress) usually.

  Sometimes, $\tau_{longitudinal}$ is the significant stress (e.g. timber)
- For deep, wide-flanged section with **large $V$ and $M$**, $\sigma_p$ may occur at the junction, ***not on the surface***. $\boxed{\tau = \frac{VQ}{It}}$

  (large $M$, $V$, $Q$, and small $t$)

  •Even though $\sigma < \sigma_{surface}$, $\tau$ is much larger than $\tau_{surface}$

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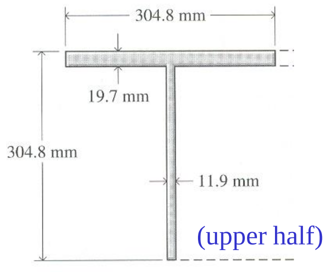
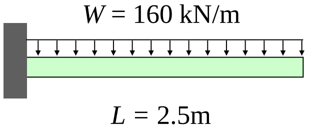
# Example Problem 7-11 (I)




Given:

- W610×145 cantilever beam under a uniformly distributed load of 160 kN/m on a span of 2.5 m as shown
  - $I = 1243(10^6)\ \text{mm}^4$
  - $S = 4079\ (10^3)\ \text{mm}^3$

Find:

- Max. normal and shearing stresses = ?

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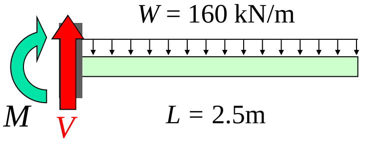
# Example Problem 7-11 (II)




- Maximum $V$ and $M$ occur on the section at the support

$$M = -\frac{wL^2}{2} = -\frac{160(2.5^2)}{2} = -500\ \text{kN}\cdot\text{m}$$

$$V = wL = 160(2.5) = 400\ \text{kN}$$

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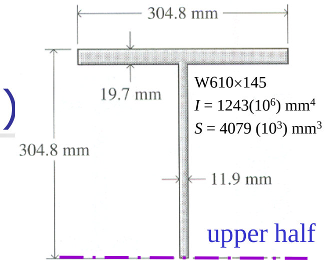
## Example Problem 7-11 (III)




- At the neutral axis

$$\sigma = 0$$

$$Q_{NA} = 304.8(19.7)(295) + 285.1(11.9)(142.6) = 2.255(10^6)\,\text{mm}^3$$

$$\tau = \frac{VQ}{It} = \frac{400(10^3)(2.255)(10^{-3})}{1243(10^{-6})(11.9)(10^{-3})}\,\text{N/m}^2 \cong 61.0\,\text{MPa}$$

- At the top surface

$$\tau = 0$$

$$\sigma = -\frac{M}{S} = -\frac{-500(10^3)}{4079(10^{-6})}\,N/m^2 \cong 122.6\,MPa\,(T)$$

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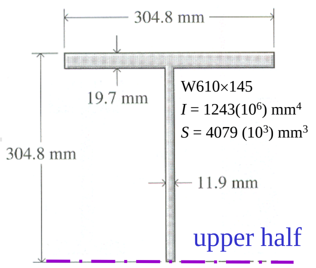
## Example Problem 7-11 (IV)



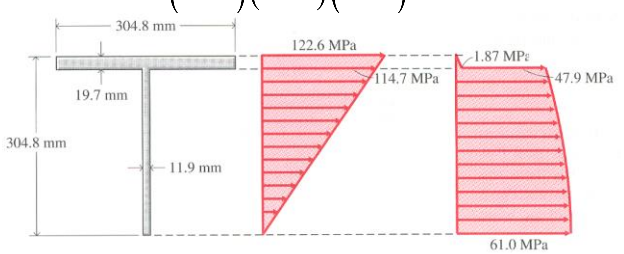

- In the web at the junction

$y = 304.8 - 19.7 = 285.1\text{mm}$

$$\sigma = -\frac{My}{I} = -\frac{-500(10^3)(0.2851)}{1243(10^{-6})}\,N/m^2 = 114.68 \cong 114.7\,MPa\,(T)$$

$$Q_J = 304.8(19.7)(295) = 1.771(10^6)\,\text{mm}^3$$

$$\tau = \frac{VQ_J}{It} = \frac{400(10^3)(1.771)(10^{-3})}{1243(10^{-6})(11.9)(10^{-3})}\,\text{N/m}^2 = 47.89 \cong 47.9\,\text{MPa}$$



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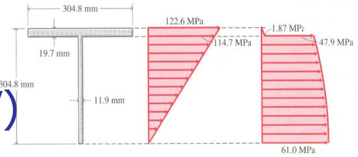
## Example Problem 7-11 (V)




- At the junction $\sigma = 114.68$ MPa, $\tau = 47.89$ MPa

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{114.68 + 0}{2} \pm \sqrt{\left(\frac{114.68 - 0}{2}\right)^2 + (-47.89)^2}$$

$$= 132.05\ MPa\ (T),\ 17.37\ MPa\ (C)$$

$\sigma = 132.1$ MPa

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{132.05 - (-17.37)}{2} \cong 74.7\ \mathrm{MPa}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{1}{2}\tan^{-1}\frac{2(-47.89)}{114.68 - 0} = -19.93°$$

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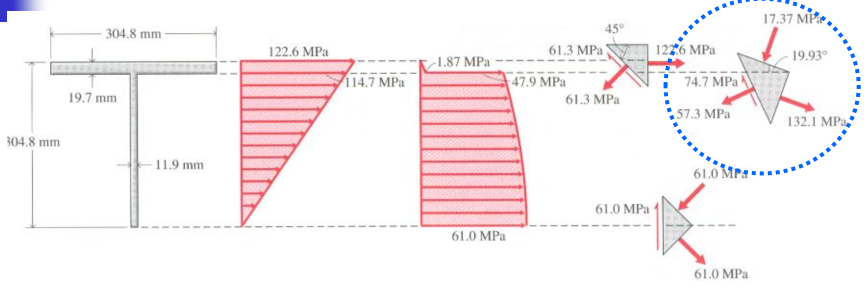
## Example Problem 7-11 (VI)




| Stress | Top Edge | Junction | Neutral Axis |
| --- | --- | --- | --- |
| $\sigma_{p1}$ | 122.6 MPa (T) | 132.1 MPa (T) | 61.0 MPa (T) |
| $\sigma_{p2}$ | 0 | 17.37 MPa (C) | 61.0 MPa (C) |

# 7 Exercises

- **7-41, 7-58, 7- 66, 7- 85, 7-187, 7-200 7-204**